

# MEASURE-GEOMETRIC LAPLACIANS FOR DISCRETE DISTRIBUTIONS

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**ABSTRACT.** In 2002 Freiberg and Zähle introduced and developed a harmonic calculus for measure-geometric Laplacians associated to continuous distributions. We show that their theory can be extended to encompass distributions with finite support and give a matrix representation for the resulting operators. In the case of a uniform discrete distribution we make use of this matrix representation to explicitly determine the eigenvalues and the eigenfunctions of the associated Laplacian.

## 1. INTRODUCTION

Motivated by the fundamental theorem of calculus, and based on the works of Feller [4] and Kac and Kreĭn [8], given an atomless Borel probability measure  $\mu$  supported on a compact subset of  $\mathbb{R}$ , Freiberg and Zähle [6] introduced a measure-geometric approach to define a first order differential operator  $\nabla^\mu$  and a second order differential operator  $\Delta^{\mu,\mu} := \nabla^\mu \circ \nabla^\mu$ , with respect to  $\mu$ . In the case that  $\mu$  is the Lebesgue measure, it was shown that  $\nabla^\mu$  coincides with the weak derivative. Moreover, a harmonic calculus for  $\Delta^{\mu,\mu}$  was developed and, when  $\mu$  is a self-similar measure supported on a Cantor set, the authors proved that the eigenvalue counting function of  $\Delta^{\mu,\mu}$  is comparable to the square-root function. In [9] for continuous measures the exact eigenvalues and eigenfunctions were obtained and it was shown that the eigenvalues do not depend on the given measure. Arzt [1, 2] has also considered the Kreĭn-Feller operator  $\Delta^{\mu,\Lambda} := \nabla^\mu \circ \nabla^\Lambda$ , where  $\mu$  denotes a continuous Borel probability measure and  $\Lambda$  denotes the Lebesgue measure, see [5, 7] for further results in this direction.

Here, we show that this framework can be extended to include purely atomic measures  $\mu$ . Unlike in the case when one has a measure with a continuous distribution function (see for instance [6, 9]), we prove that the operators  $\nabla^\mu$  and  $\Delta^{\mu,\mu}$  are no longer symmetric. To circumvent this problem, we consider the operator  $\nabla^\mu$ , its adjoint  $(\nabla^\mu)^*$  and define the  $\mu$ -Laplacian to be  $\Delta^\mu = -(\nabla^\mu)^* \circ \nabla^\mu$ . We give matrix representations for these operators, noting that they coincide with the normalised Laplacian matrix of a cycle graph [3] and resemble a discretisation of a one-dimensional Laplacian on a non-uniform grid [14]. Further, we discuss properties of the eigenvalues and eigenfunctions of the operator  $\Delta^\mu$ . In particular, we show that the eigenfunctions for distributions with finite support are not necessarily of the form  $f_\kappa^\mu(\cdot) := \sin(\pi\kappa F_\mu(\cdot))$  or  $g_\kappa^\mu(\cdot) := \cos(\pi\kappa F_\mu(\cdot))$ , for  $\kappa \in \mathbb{R} \setminus \{0\}$  and where  $F_\mu$  denotes the distribution function of  $\mu$ . This differs from the case of continuous distributions, see [6, 9, 15]. Additionally, in the case that  $\mu$  is a uniform discrete probability distribution we explicitly determine the eigenvalues and eigenfunctions of  $\Delta^\mu$ .

**Outline.** In Section 2 we present necessary definitions and basic properties of  $\nabla^\mu$ ,  $(\nabla^\mu)^*$  and  $\Delta^\mu$  and give matrix representations for these operators. In Section 3 we prove general results concerning the spectral properties of  $\Delta^\mu$ . We conclude with Section 4, where explicit computations are carried out when  $\mu$  is a uniform discrete probability distribution.

2. DEFINITIONS AND ANALYTIC PROPERTIES OF  $\Delta^\mu$ 

Set  $\mathcal{I} := [0, 1]$  and let  $\delta_z$  denote the Dirac-measure at  $z$ , for some fixed  $z \in \mathcal{I}$ . Let  $\mu$  denote the probability measure  $\mu := \sum_{i=1}^N \alpha_i \delta_{z_i}$ , where  $N \in \mathbb{N}$ ,  $0 \leq z_1 < z_2 < \dots < z_N < 1$  and  $\alpha_i > 0$ , for  $i \in \{1, \dots, N\}$ . We denote the set of real-valued square-integrable functions with domain  $\mathcal{I}$  by  $\mathfrak{L}_\mu^2 = \mathfrak{L}_\mu^2(\mathcal{I})$ , we define  $\mathcal{N}_\mu(\mathcal{I})$  to be set of  $\mathfrak{L}_\mu^2$ -functions which are constant zero  $\mu$ -almost everywhere, and we let  $L_\mu^2 = L_\mu^2(\mathcal{I}) := \mathfrak{L}_\mu^2(\mathcal{I}) \setminus \mathcal{N}_\mu(\mathcal{I})$ . The latter space is a finite-dimensional inner product space with inner product  $\langle \cdot, \cdot \rangle$  given by

$$\langle f, g \rangle = \langle f, g \rangle_\mu := \sum_{i=1}^N \alpha_i f(z_i) g(z_i).$$

We define the set of  $\mu$ -differentiable functions on  $\mathcal{I}$  with periodic boundary conditions by

$$\begin{aligned} \mathcal{D}_\mu^1 = \mathcal{D}_\mu^1(\mathcal{I}) := \left\{ f \in \mathfrak{L}_\mu^2(\mu) : \text{there exists } f' \in L_\mu^2 \text{ such that } f(0) = f(1) \text{ and} \right. \\ \left. f(x) = f(0) + \int_{[0,x)} f' d\mu \text{ for all } x \in \mathcal{I} \right\}, \end{aligned} \quad (1)$$

where we understand  $[0, 0) = \emptyset$ . Note that, the function  $f'$  defined in (1) is unique in  $L_\mu^2$ . Since  $f(0) = f(1) = f(0) + \int_{[0,1)} f' d\mu$ , it follows that

$$\int_{[0,1)} f' d\mu = 0. \quad (2)$$

For  $f \in \mathcal{D}_\mu^1$  and  $f'$  as in (1), the operator  $\nabla^\mu : \mathcal{D}_\mu^1 \rightarrow L_\mu^2$  defined by  $\nabla^\mu f := f'$  is called the  $\mu$ -derivative. Linearity of the integral yields that  $\nabla^\mu$  is linear on  $\mathcal{D}_\mu^1$ . As  $\mu$  is a linear combination of Dirac measures, we can reformulate the defining equation of  $\nabla^\mu f$  given in (1) by

$$f(x) = f(0) + \sum_{\substack{i \in \{1, \dots, N\} \\ z_i < x}} \alpha_i \nabla^\mu f(z_i), \quad (3)$$

where  $f \in \mathcal{D}_\mu^1$  and  $x \in \mathcal{I}$ . Thus, if  $\mu$  is a Dirac measure, that is  $N = 1$ , then (2) becomes  $\alpha_1 \nabla^\mu f(z_1) = 0$ . Hence, from (3), it follows that  $\mathcal{D}_\mu^1$  is the set of constant functions. In other words, the operator  $\nabla^\mu$  is the null-operator, and so, from here on we assume that  $N \geq 2$ .

The periodic boundary conditions and (3) together imply that a function  $f \in \mathcal{D}_\mu^1$  is piecewise constant; namely,  $f|_{[0, z_1] \cup (z_N, 1]}$  and  $f|_{[z_i, z_{i+1}]}$  are constant, for all  $i \in \{1, \dots, N-1\}$ . Therefore,  $f$  is uniquely determined by the vector  $(f(z_1), \dots, f(z_N))^\top$ , and thus, there exists an  $N \times N$ -matrix  $A$  with

$$A(f(z_1), \dots, f(z_N))^\top = (\nabla^\mu f(z_1), \dots, \nabla^\mu f(z_N))^\top.$$

From (3) and the fact that  $f(1) = f(0) = f(z_1)$ , we have that

$$\nabla^\mu f(z_N) = \frac{f(z_1) - f(z_N)}{\alpha_N} \quad \text{and} \quad \nabla^\mu f(z_n) = \frac{f(z_{n+1}) - f(z_n)}{\alpha_n},$$

for  $n \in \{1, \dots, N-1\}$ , and hence,

$$A = \begin{pmatrix} -\alpha_1^{-1} & \alpha_1^{-1} & 0 & \dots & 0 & 0 & 0 \\ 0 & -\alpha_2^{-1} & \alpha_2^{-1} & \dots & 0 & 0 & 0 \\ 0 & 0 & -\alpha_3^{-1} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -\alpha_{N-2}^{-1} & \alpha_{N-2}^{-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & -\alpha_{N-1}^{-1} & \alpha_{N-1}^{-1} \\ \alpha_N^{-1} & 0 & 0 & \dots & 0 & 0 & -\alpha_N^{-1} \end{pmatrix}.$$

Since  $\sum_{i=1}^N \alpha_i g^2(z_i) < \infty$ , for all  $g \in \mathcal{D}_\mu^1$ , there exists a natural embedding  $\pi: \mathcal{D}_\mu^1 \rightarrow L_\mu^2$ . In fact, from the matrix representation given above it follows that  $\pi(\mathcal{D}_\mu^1) = L_\mu^2$ . In other words, every equivalence class of  $L_\mu^2$  has a  $\mu$ -differentiable representative, and so, from here on we will not distinguish between  $\mathcal{D}_\mu^1$  and  $\pi(\mathcal{D}_\mu^1)$ .

Notice,  $A$  is not self-adjoint, and that  $A^2$  is self-adjoint if and only if  $N = 2$  and  $\alpha_1 = \alpha_2$ . Hence, the operators  $\nabla^\mu$  and  $\Delta^\mu := \nabla^\mu \circ \nabla^\mu$  are not in general self-adjoint. To obtain a self-adjoint operator we follow the program of Kigami [10, 11, 12], and Kigami and Lapidus [13], and use the bilinear form  $\mathcal{E}$  defined by  $\mathcal{E}(f, g) = \mathcal{E}^\mu(f, g) := \langle \nabla^\mu f, \nabla^\mu g \rangle$ , for  $f, g \in \mathcal{D}_\mu^1$ . We refer to  $\mathcal{E}$  as the  $\mu$ -energy form.

**Theorem 2.1.** *The  $\mu$ -energy form  $\mathcal{E}$  is a Dirichlet form.*

*Proof.* The  $\mu$ -energy form is bilinear since the inner product is bilinear,  $\nabla^\mu$  is linear and every equivalence class of  $L_\mu^2$  has a  $\mu$ -differentiable representative. The symmetry and the non-negativity of  $\mathcal{E}$  follow from the properties of the inner product. For every  $f \in \mathcal{D}_\mu^1$ , the function  $\hat{f}: I \rightarrow \mathbb{R}$ , defined by  $\hat{f}(x) := \min(\max(f(x), 0), 1)$ , belongs to  $\mathcal{D}_\mu^1$ , and as

$$|\hat{f}(z_{i+1}) - \hat{f}(z_i)| \leq |f(z_{i+1}) - f(z_i)| \quad \text{and} \quad |\hat{f}(z_1) - \hat{f}(z_N)| \leq |f(z_1) - f(z_N)|,$$

it follows that  $\mathcal{E}(\hat{f}, \hat{f}) \leq \mathcal{E}(f, f)$ . The properties of  $\langle \cdot, \cdot \rangle$  and  $\mathcal{E}$  yield that  $\mathcal{D}_\mu^1$  equipped with  $\langle \cdot, \cdot \rangle_\mathcal{E} := \langle \cdot, \cdot \rangle + \mathcal{E}(\cdot, \cdot)$  is an inner product space. Now, for every Cauchy sequence  $(f_n)_{n \in \mathbb{N}}$  in  $(\mathcal{D}_\mu^1, \langle \cdot, \cdot \rangle_\mathcal{E})$ , we have that both  $(f_n)_{n \in \mathbb{N}}$  and  $(\nabla^\mu f_n)_{n \in \mathbb{N}}$  are Cauchy-sequences in  $L_\mu^2$ . Hence, there exist  $\tilde{f}_0, \tilde{f}_1 \in L_\mu^2$  with  $\lim_{n \rightarrow \infty} \|f_n - \tilde{f}_0\| = 0$  and  $\lim_{n \rightarrow \infty} \|\nabla^\mu f_n - \tilde{f}_1\| = 0$ , where  $\|f\|^2 := \langle f, f \rangle$ , for  $f \in L_\mu^2$ . Since  $\nabla^\mu$  is linear, it is continuous, and so  $\nabla^\mu \tilde{f}_0 = \tilde{f}_1$ . This implies that  $\lim_{n \rightarrow \infty} f_n = \tilde{f}_0 \in \mathcal{D}_\mu^1$ , with respect to  $\langle \cdot, \cdot \rangle_\mathcal{E}$ . ■

We say that  $f \in \mathcal{D}_\mu^1$  belongs to  $\mathcal{D}_\mu^2 = \mathcal{D}_\mu^2(I)$ , if there exists a  $h \in L_\mu^2$ , necessarily unique, such that  $\mathcal{E}(f, g) = -\langle h, g \rangle$ , for all  $g \in \mathcal{D}_\mu^1$ . We define the  $\mu$ -Laplacian to be the operator  $\Delta^\mu: \mathcal{D}_\mu^2 \rightarrow L_\mu^2$  given by  $\Delta^\mu f := h$ . Indeed, for an arbitrary  $g \in \mathcal{D}_\mu^1$ , we observe that

$$\langle \nabla^\mu f, \nabla^\mu g \rangle = -\langle \Delta^\mu f, g \rangle, \quad \text{and thus,} \quad \Delta^\mu = -(\nabla^\mu)^* \circ \nabla^\mu. \quad (4)$$

With this at hand, we conclude that  $B := -A^\top A$  is a matrix representation of  $\Delta^\mu$ ; in fact, for  $N = 2$ ,

$$B = \begin{pmatrix} -\alpha_1^{-2} - \alpha_2^{-2} & \alpha_1^{-2} + \alpha_2^{-2} \\ \alpha_1^{-2} + \alpha_2^{-2} & -\alpha_1^{-2} - \alpha_2^{-2} \end{pmatrix}, \quad (5)$$

and, for  $N \geq 3$ , we have that  $B$  is the  $N \times N$ -matrix

$$\begin{pmatrix} -\alpha_N^{-2} - \alpha_1^{-2} & \alpha_1^{-2} & 0 & \cdots & 0 & 0 & \alpha_N^{-2} \\ \alpha_1^{-2} & -\alpha_1^{-2} - \alpha_2^{-2} & \alpha_2^{-2} & \cdots & 0 & 0 & 0 \\ 0 & \alpha_2^{-2} & -\alpha_2^{-2} - \alpha_3^{-2} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\alpha_{N-3}^{-2} - \alpha_{N-2}^{-2} & \alpha_{N-2}^{-2} & 0 \\ 0 & 0 & 0 & \cdots & \alpha_{N-2}^{-2} & -\alpha_{N-2}^{-2} - \alpha_{N-1}^{-2} & \alpha_{N-1}^{-2} \\ \alpha_N^{-2} & 0 & 0 & \cdots & 0 & \alpha_{N-1}^{-2} & -\alpha_{N-1}^{-2} - \alpha_N^{-2} \end{pmatrix}.$$

**Theorem 2.2.** *The operator  $\Delta^\mu$  is linear, self-adjoint and non-positive.*

*Proof.* Linearity follows from the linearity of  $\nabla^\mu$  and the bilinearity of  $\mathcal{E}$ . Self-adjointness is a consequence of the symmetry of  $\mathcal{E}$ . Using (4), we have  $\langle \Delta^\mu f, f \rangle = -\langle \nabla^\mu f, \nabla^\mu f \rangle \leq 0$ , and hence,  $\Delta^\mu$  is non-positive. ■

The above demonstrates that one can view the operator  $\Delta^\mu$  as a Laplacian matrix of a weighted cycle graph, see [3]. Further, the matrix  $B$  resembles the matrices appearing in the finite difference methods used in numerical analysis of ordinary differential equations<sup>1</sup>, see [14]. We also observe that the matrix representation of the operator  $\Delta^\mu$  only depends on the order and the weighting of the atoms of  $\mu$  and is independent on the distances between  $z_i$  and  $z_j$ , for all  $i, j \in \{1, \dots, N\}$ .

### 3. SPECTRAL PROPERTIES OF $\Delta^\mu$

By definition of the matrix  $B$ , to find the eigenvalues and eigenfunctions of  $\Delta^\mu$ , it suffices to compute the eigenvalues and eigenvectors of  $B$ .

**Proposition 3.1.** *If  $\lambda$  is an eigenvalue of  $\Delta^\mu$ , then  $\lambda \in \mathbb{R}$  and  $2 \min_{i \in \{1, \dots, N\}} B_{i,i} \leq \lambda \leq 0$ .*

*Proof.* Theorem 2.2 together with the fact that  $B$  is self-adjoint with entries in  $\mathbb{R}$  yields that all eigenvalues are non-positive real numbers. Since the spectral norm is bounded by the column-sum norm, given an eigenvalue  $\lambda$  of  $B$ , we have that  $|\lambda| \leq 2 \max_{i \in \{1, \dots, N\}} |B_{i,i}|$ . ■

**Proposition 3.2.** *The operator  $\Delta^\mu$  has a simple eigenvalue at  $\lambda = 0$  where the corresponding eigenfunction is the constant function with value 1.*

*Proof.* A direct calculation reveals that  $B(1, \dots, 1)^\top = (0, \dots, 0)^\top$ , and hence, we have that  $\lambda = 0$  is an eigenvalue of  $\Delta^\mu$  where the constant function with value 1 is the corresponding eigenfunction. A row reduced echelon form of  $A$  is an upper triangular matrix with a single zero on the diagonal, and so  $\text{rank}(A) = N - 1$ . Combining this with the fact that  $\text{rank}(B) = \text{rank}(A^\top A) = \text{rank}(A)$  it follows that the eigenvalue  $\lambda = 0$  of  $B$  is simple. ■

We observe that, for  $N = 2$ , the eigenvalues of  $\Delta^\mu$  are  $\lambda_0 = 0$  and  $\lambda_1 = -2(\alpha_1^{-2} + \alpha_2^{-2})$  with corresponding eigenfunctions,

$$f_0(x) = 1, \text{ for all } x \in [0, 1], \quad \text{and} \quad f_1(x) = \begin{cases} 1 & \text{for } x \in [0, z_1] \cup (z_2, 1], \\ -1 & \text{otherwise.} \end{cases}$$

This follows since  $\lambda_0$  and  $\lambda_1$  are eigenvalues of the matrix  $B$ , given in (5), with corresponding eigenvectors  $v^{(0)} = (1, 1)^\top$  and  $v^{(1)} = (1, -1)^\top$ , respectively. In particular, we see that, in this case, the lower bound in Theorem 3.1 is sharp.

Different to the case of continuous distributions, the eigenfunctions for distributions with finite support are not necessarily of the form  $f_k^\mu(x) = \sin(\pi \kappa F_\mu(x))$  or  $g_k^\mu(x) = \cos(\pi \kappa F_\mu(x))$ , for  $x \in [0, 1]$  and  $\kappa \in \mathbb{R} \setminus \{0\}$ , where  $F_\mu$  denotes the distributions function of  $\mu$ , which we now address in the following paragraph.

Let  $m_1$  and  $m_2$  denote two positive real numbers with  $3m_1 + 3m_2 = 1$ , and set  $r = m_2/m_1$ . Consider the discrete distribution  $\mu = \sum_{i=1}^6 \alpha_i \delta_{z_i}$ , where  $0 < z_1 < z_2 < \dots < z_6 < 1$ ,  $\alpha_1 = \alpha_3 = \alpha_5 = m_1$  and  $\alpha_2 = \alpha_4 = \alpha_6 = m_2$ . A direct calculation shows that the eigenvalues for the matrix representation of  $\Delta^\mu$  are

$$\begin{aligned} \lambda_0 &= 0, & \lambda_1 &= \lambda_5 = -(m_1^{-2} + m_2^{-2}) + \sqrt{m_1^{-4} + m_2^{-4} - m_1^{-2}m_2^{-2}}, \\ \lambda_3 &= -2(m_1^{-2} + m_2^{-2}), & \lambda_2 &= \lambda_4 = -(m_1^{-2} + m_2^{-2}) - \sqrt{m_1^{-4} + m_2^{-4} - m_1^{-2}m_2^{-2}}, \end{aligned}$$

with corresponding eigenvectors

$$v^{(0)} = (1, 1, 1, 1, 1, 1)^\top,$$

<sup>1</sup> We would like to thank Paul Choboter, Maik Gröger, Jens Rademacher and Alfred Schmidt for bringing this to our attention.

$$\begin{aligned}
v^{(1)} &= (r^2, \sqrt{1+r^4-r^2}, 1-r^2, -\sqrt{1+r^4-r^2}, -1, 0)^\top, \\
v^{(2)} &= (r^2, -\sqrt{1+r^4-r^2}, 1-r^2, \sqrt{1+r^4-r^2}, -1, 0)^\top, \\
v^{(3)} &= (1, -1, 1, -1, 1, -1)^\top, \\
v^{(4)} &= (\sqrt{1+r^4-r^2}, 1-r^2, -\sqrt{1+r^4-r^2}, r^2, 0, -1)^\top, \\
v^{(5)} &= (\sqrt{1+r^4-r^2}, r^2-1, -\sqrt{1+r^4-r^2}, -r^2, 0, 1)^\top.
\end{aligned}$$

For the case  $m_1 = 1/4$  and  $m_2 = 1/12$ , in Figure 1 the corresponding eigenfunctions of the operator  $\Delta^\mu$  are sketched. Interestingly, for the eigenvalues with multiplicity two, namely  $\lambda_1 = \lambda_5$  and  $\lambda_2 = \lambda_4$ , we notice that the sets of tuples

$$S_{1,5} := \{(v_1^{(1)}, v_1^{(5)}), \dots, (v_6^{(1)}, v_6^{(5)})\} \quad \text{and} \quad S_{2,4} := \{(v_1^{(2)}, v_1^{(4)}), \dots, (v_6^{(2)}, v_6^{(4)})\}$$

determine the same ellipse, namely,

$$(5329 + 595\sqrt{73})(x^2 + y^2 - 1) = -7(595 + 73\sqrt{73})xy,$$

which is non-axisymmetric, see Figure 2 and compare with Theorem 4.1 and Corollary 4.2, where the analogous set of tuples lie on the unit circle. This latter property demonstrates that the eigenspace for  $\lambda_1 = \lambda_5$  (or  $\lambda_2 = \lambda_4$ ) is not spanned by  $\{f_{\kappa_1}^\mu, g_{\kappa_2}^\mu\}$  for any  $\kappa_1, \kappa_2 \in \mathbb{R}$ . Moreover, in this explicit case, setting

$$w^{(\kappa)} := (\sin(\pi\kappa F_\mu(z_1)), \dots, \sin(\pi\kappa F_\mu(z_6)))^\top, \quad u^{(\kappa)} := (\cos(\pi\kappa F_\mu(z_1)\pi), \dots, \cos(\pi\kappa F_\mu(z_6)\pi))^\top,$$

a direct calculation shows that  $Bw^{(\kappa)} \neq \lambda_i w^{(\kappa)}$  and  $Bu^{(\kappa)} \neq \lambda_i u^{(\kappa)}$ , for all  $\kappa \in \mathbb{R} \setminus \{0\}$  and  $i \in \{0, \dots, 5\}$ . This latter result also holds true when replacing  $F_\mu(z_i)$  by  $\tilde{F}_\mu(z_i) := F_\mu(z_i - \varepsilon)$ , for a fixed  $\varepsilon \in (0, \min\{z_1, \min\{z_{i+1} - z_i : i \in \{1, \dots, 5\}\})\}$ .

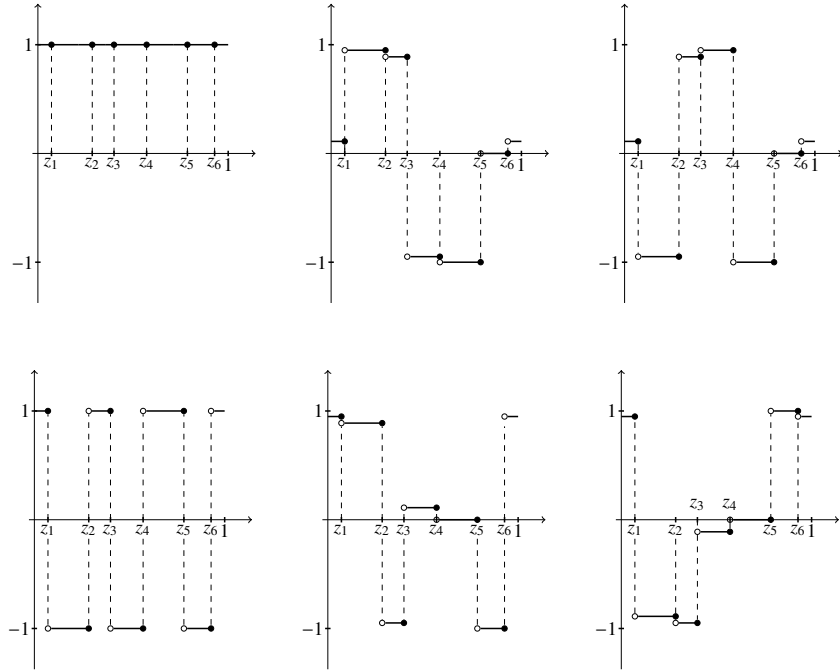


FIGURE 1. Eigenfunctions  $f_0, f_1, f_2, f_3, f_4$  and  $f_5$  of  $\Delta^\mu$  for  $\mu = \sum_{i=1}^6 \alpha_i \delta_{z_i}$  with  $\alpha_1 = \alpha_3 = \alpha_5 = 1/4$  and  $\alpha_2 = \alpha_4 = \alpha_6 = 1/12$ . Compare with Figure 4.

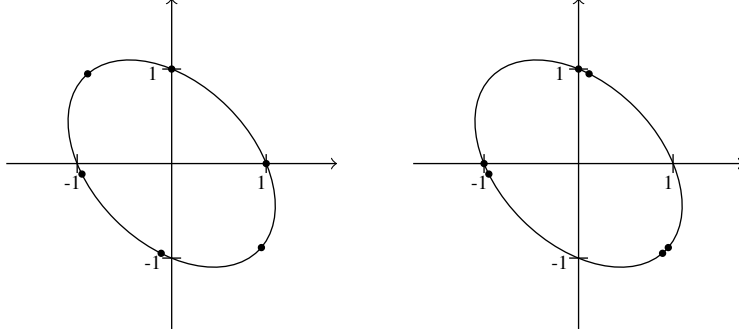


FIGURE 2. Point plot of  $S_{1,5}$  (left) and point plot of  $S_{2,4}$  (right) together with the curve given by  $(5329 + 595\sqrt{73})(x^2 + y^2 - 1) = -7(595 + 73\sqrt{73})xy$ .

#### 4. UNIFORM DISCRETE PROBABILITY DISTRIBUTIONS

Here, we consider the case when  $\mu$  is a uniform discrete probability distribution with  $N \geq 3$ , namely  $\alpha_i = N^{-1}$  for all  $i \in \{1, \dots, N\}$ , in which case,

$$B = \begin{pmatrix} -2N^{-2} & N^{-2} & 0 & \dots & 0 & 0 & N^{-2} \\ N^{-2} & -2N^{-2} & N^{-2} & \dots & 0 & 0 & 0 \\ 0 & N^{-2} & -2N^{-2} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -2N^{-2} & N^{-2} & 0 \\ 0 & 0 & 0 & \dots & N^{-2} & -2N^{-2} & N^{-2} \\ N^{-2} & 0 & 0 & \dots & 0 & N^{-2} & -2N^{-2} \end{pmatrix}.$$

The following theorem reveals the spectrum of this matrix, and in the case that  $N$  is even, the result gives a second example for which the lower bound in Theorem 3.1 is sharp.

**Theorem 4.1.** *The eigenvalues of the matrix  $B$  given directly above are of the form*

$$\lambda_l = -2N^{-2} + 2N^{-2} \cos(2\pi l/N)$$

*with corresponding eigenvectors*

$$v^{(l)} = \left( 1, \exp(2\pi i l/N), \exp(2\pi i 2l/N), \dots, \exp(2\pi i (N-1)l/N) \right)^\top,$$

for  $l \in \{0, \dots, N-1\}$ .

*Proof.* Set  $m_1 = -2N^{-2}$ ,  $m_2 = m_N = N^{-2}$  and  $m_i = 0$ , for  $i \in \{3, \dots, N-1\}$ . The eigenvalue equation  $Bv = \lambda v$  can be formulated as a system of  $N$  difference equations of the form

$$\sum_{k=1}^{N-j} m_k v_{k+j} + \sum_{k=N-j+1}^N m_k v_{k-N+j} = \lambda v_{j+1}, \quad (6)$$

where  $j \in \{0, \dots, N-1\}$  and  $v = (v_1, \dots, v_N)$ . To obtain the eigenvalue  $\lambda_l$ , we follow the ansatz  $v_k^{(l)} = \varphi_l^{k-1}$ , where  $\varphi_l^k := \exp(2\pi i k l/N)$ , for  $k, l \in \{0, \dots, N-1\}$ . Substituting this into (6), and using the facts that  $\varphi_l^{-N} = \varphi_l^0 = 1$  and  $\varphi_l^j \neq 0$ , for all  $j, l \in \{0, \dots, N-1\}$ , we obtain that  $\lambda_l = \sum_{k=1}^N m_k \varphi_l^{k-1}$ . Hence, for  $l \in \{0, \dots, N-1\}$ , we have that  $Bv^{(l)} = \lambda_l v^{(l)}$  and

$$\begin{aligned} \lambda_l &= -2N^{-2} + N^{-2} \exp(2\pi i l/N) + N^{-2} \exp(2\pi i l(N-1)/N) \\ &= -2N^{-2} + 2N^{-2} \cos(2\pi l/N). \end{aligned}$$

■

**Corollary 4.2.** *The eigenvalues of the operator  $\Delta^\mu$  are  $\lambda_l = -2N^{-2} + 2N^{-2} \cos(2\pi l/N)$ , for  $l \in \{0, \dots, N-1\}$ , with corresponding eigenfunctions  $f_l \in \mathcal{D}_\mu^1$ , where*

(1)  $f_0$  is the constant function with value 1,

and, for  $j \in \{1, \dots, N-1\}$ ,

- (2)  $f_l|_{[0, z_1] \cup (z_N, 1]} = 0$  and  $f_l|_{(z_j, z_{j+1}]} = \text{Im}(\exp(2\pi i j l/N))$ , for  $0 < l < N/2$ , and  
 (3)  $f_l|_{[0, z_1] \cup (z_N, 1]} = 1$  and  $f_l|_{(z_j, z_{j+1}]} = \text{Re}(\exp(2\pi i j l/N))$ , for  $N/2 \leq l \leq N-1$ .

(See Figures 3 and 4 below.)

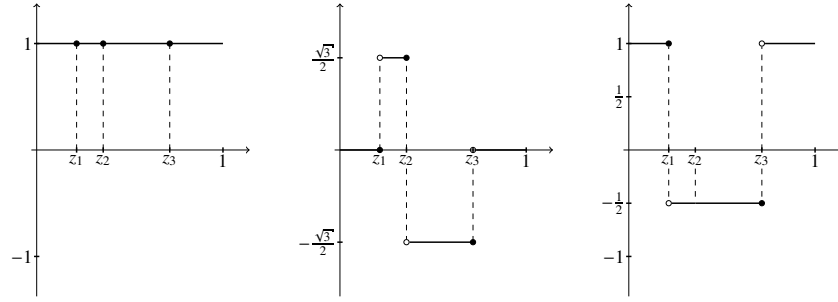


FIGURE 3. Eigenfunctions  $f_0$ ,  $f_1$  and  $f_2$  of  $\Delta^\mu$  with corresponding eigenvalues  $\lambda_0 = 0$ ,  $\lambda_1 = -1/3$  and  $\lambda_2 = -1/3$ , for  $\mu$  a uniform discrete probability distribution with  $N = 3$ .

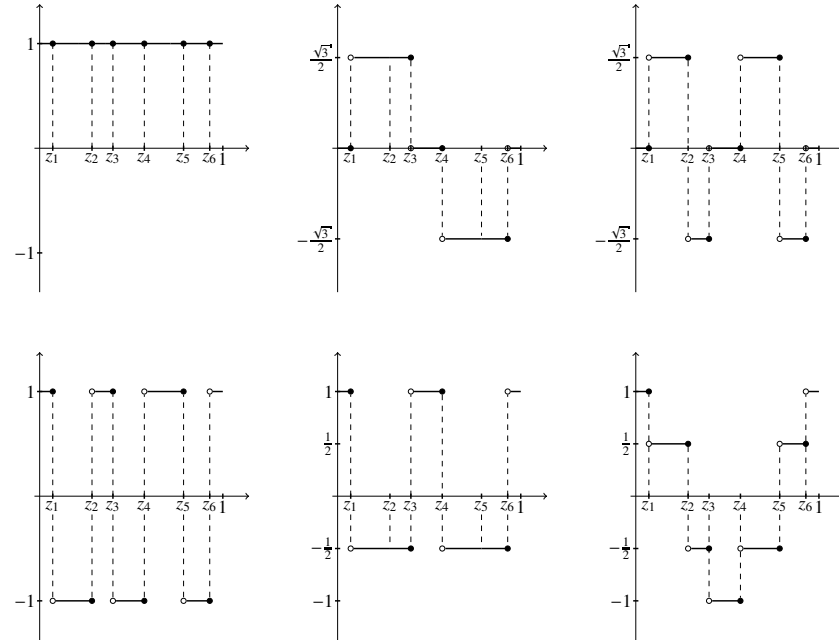


FIGURE 4. Eigenfunctions  $f_0$ ,  $f_1$ ,  $f_2$ ,  $f_3$ ,  $f_4$  and  $f_5$  of  $\Delta^\mu$  with corresponding eigenvalues  $\lambda_0 = 0$ ,  $\lambda_1 = -1/36$ ,  $\lambda_2 = -1/12$ ,  $\lambda_3 = -1/9$ ,  $\lambda_4 = -1/12$  and  $\lambda_5 = -1/36$ , for  $\mu$  a uniform discrete probability distribution with  $N = 6$ . Compare with Figure 1.

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